Journal of Engineering Physics and Thermophysics, Vol. 74, No. 1, 2001

CALCULATION OF AVERAGED PARAMETERS OF INHOMOGENEOUS MEDIA WITH INCLUSIONS OF VARIABLE PROPERTIES

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We consider inhomogeneous materials with inclusions having variable parameters. Expressing the polarization energy of the inclusions as a function of its parameter and using the law of conservation of energy, we obtain an integral relation allowing one to derive new formulas to calculate the mean dielectric permeability of a mixture with inclusions of different shapes.

Inhomogeneous media that include composite, granular, and fibrous materials and various alloys and compounds are widely used in all fields of technology. The presently available natural and artificial inhomogeneous materials are about five million in number, and their number increases yearly by about a hundred thousand. A multitude of research works are devoted to the study of electromagnetic, diffusional, and mechanical properties of inhomogeneous materials under conditions of application of various physical fields. A great number of them, motivated by the needs of technology, were published from 1960 to 1990 (see review papers and monographs [1–5]).

The theory of inhomogeneous materials is mainly aimed at determining the averaged (effective) parameters of inhomogeneous media. To solve this very formidable problem, it is necessary to calculate the physical fields in all the components of the inhomogeneous material. This requires detailed information on the structure, orientation, and distribution of all the components of the inhomogeneous material. Solution of the system of equations of state for an inhomogeneous medium, especially in the presence of a flow process, involves great difficulties and can be obtained only for the simplest structures. This explains the multitude of models, methods, and formulas for determining the averaged parameters of inhomogeneous media that have been suggested in the period of more than a century of study of the physical properties of the materials under consideration [1–5].

The theory of inhomogeneous materials (the theory of mixtures) is being developed along two major lines: refinement of the available formulas and provision of new models, methods, and formulas for determining effective parameters of inhomogeneous materials. Known investigations are integrated into the following main groups: semiempirical methods, methods of an effective medium; integral and asymptotic methods and construction of a function; geometric simulation of the structure of an inhomogeneous material [5]. We will consider these methods for a system of dielectric in a dielectric.

In the semiempirical method, the system of equations of state for a medium is supplemented with an experimentally derived relation between the parameters, fields, and concentrations of the components of the inhomogeneous material. For individual inhomogeneous systems these data are approximated, while the components entering into these expressions are determined experimentally. The drawbacks of the method are its insufficient accuracy and the impossibility of determining the limits of applicability of an averaged parameter.

In the method of an effective medium, an inhomogeneous material is modeled by an arbitrarily selected particle surrounded by a medium with unknown (averaged) properties [2–5]. In using various ap-

State Engineering University of Armenia, Erevan, Armenia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 74, No. 1, pp. 92–98, January–February, 2001. Original article submitted December 3, 1999; revision submitted April 14, 2000.

proaches and mathematical methods, identical expressions were obtained for the averaged parameters from various models. As a result, the same formula for the averaged parameters was obtained by Maxwell and Garnet, Lorenz and Lorentz, Wiener, and Wagner [1–3]. Another formula for the mean dielectric permeability $\langle\varepsilon\rangle$, given in [3–5], was first derived in 1935 and afterwards derived repeatedly by Odelevskii and Batcher:

$$\langle\varepsilon\rangle = \varepsilon_1 \left\{ \left[(3f_1 - 1) + (3f_2 - 1)\frac{\varepsilon_2}{\varepsilon_1} \right] / 4 + \sqrt{\left[(3f_1 - 1) + (3f_2 - 1)\frac{\varepsilon_2}{\varepsilon_1} \right]^2 / 16 + \varepsilon_2 / 2\varepsilon_1} \right\}; \quad (1)$$

It is known as the Kondorskii–Odelevskii formula [2, 5]. The main drawback of the effective-medium method is that the dielectric permeability is negative at $\nu = \varepsilon_2/\varepsilon_1 = 0$ and for $f_1 < 0.3$, which contradicts the physical meaning; when $\nu < 10^{-2}$, there is a discrepancy with experimental data. And the Maxwell–Lorenz–Wagner formula is applicable for $f_2 << 1$.

The integral method was suggested by Bruggeman [3] and thereafter was reused by T. Hanai and V. V. Skorokhod for describing an imperfect and inhomogeneous dielectric [5]. Despite its internal inconsistency, the Bruggeman–Hanai formula is in good agreement with experiment. Analysis of it shows that at zero dielectric permeability of the dispersion medium ε_1 the averaged parameter $\langle\varepsilon\rangle$ of the inhomogeneous material is equal to zero for any concentration of the disperse phase. This result is incorrect and limits the applicability of this method [5].

The asymptotic method (L. D. Landau and E. M. Lifshits, V. V. Novikov) uses a successive approximation, where the suggested mathematical model is investigated and supplemented until the best possible agreement is obtained between the theoretical and experimental data [5]. A significant drawback of the method is that it disregards the structure of the mixture and that the expression for the averaged parameter is of the same form for inhomogeneous media with inclusions of different shapes.

Recently, fundamental investigations appeared in the theory of inhomogeneous media that are devoted to the problem of averaging inhomogeneous materials. But averaging can be carried out only for inhomogeneous media of periodic structure. For a dielectric inhomogeneous medium of periodic structure with prescribed static sources $f(x)$ in the equation

$$-\frac{\partial}{\partial x_i}(\varepsilon_{ij}(x))\frac{\partial\varphi}{\partial x_j} = f(x) \quad (2)$$

the tensor of the dielectric permeability $\varepsilon_{ij}(x)$, which expresses the physical properties of the inhomogeneous material, is the symmetric matrix $\varepsilon_{ij}(x) = \varepsilon_{ji}(x)$ and is periodic over the spatial coordinate $\vec{r}$. If the period is very short in comparison with other macroscopic parameters, the solution of the potential φ of Eq. (2) is nearly the same as the corresponding solution for the averaged (homogeneous) material with a constant matrix $\langle\varepsilon\rangle$. The method of averaging proves the existence of the matrix $\langle\varepsilon\rangle$, which is independent of $\vec{r}$ [6].

Based on the definition of Lorenz averaging

$$\langle\varepsilon\rangle = \frac{1}{V_0}\int_{V_0} \varepsilon dV \quad (3)$$

it is proved in [6] that the averaged vectors of the electromagnetic field $\langle\vec{E}\rangle$, $\langle\vec{D}\rangle$, $\langle\vec{B}\rangle$, $\langle\vec{H}\rangle$, and $\langle\vec{\delta}\rangle$ satisfy the system of Maxwell equations. In particular, we write

$$\langle\vec{D}\rangle = \langle\varepsilon\rangle\langle\vec{E}\rangle, \quad \langle\vec{B}\rangle = \langle\mu\rangle\langle\vec{H}\rangle. \quad (4)$$

For isotropic dielectrics there is the linear dependence (4) in which simultaneous vanishing of $\langle \vec{D} \rangle$ and $\langle \vec{E} \rangle$ is valid only in dielectrics that are homogeneous in their physical properties. In inhomogeneous materials, $\langle \vec{D} \rangle$ can differ from zero also at $\langle \vec{E} \rangle = 0$. This is due to the gradients of thermodynamic quantities that vary along the body, which, however, are very small and actually are not important for any of the phenomena, which was proved in [6] in the form of relation (4).

In analyzing existing investigations, one proceeds from general requirements that are to be satisfied. These are the equivalence of the suggested model to the actual inhomogeneous medium, the possibility of obtaining physically correct results in limiting cases, the absence of internal inconsistencies in the theoretical scheme, and satisfactory correspondence of the results of a theoretical calculation to experimental data in a wide range of determining parameters [5].

In the present work, based on the theory of averaging of inhomogeneous materials a method is suggested for calculating the averaged parameters of inhomogeneous media; the method satisfies the above-indicated general requirements. For the first time we investigate inhomogeneous media, the parameters of the inclusions of which are variable quantities and can vary from unity to infinity. This can be done using the property of a potential field that allows one to express the energy of an electrostatic field as a function of both the field intensity (the parameters of the medium are constant) and the properties of the medium (the parameters of the medium are variable). The suggested method is general and is applied to currently studied mixtures with inclusions of any geometric shape. Investigations are carried out for a system of dielectric in a dielectric where there is a simple electrical layer at the interface between inhomogeneities. For this two-component medium with constant parameters of the dielectric permeabilities of the inclusion ε_2 and the dispersion medium ε_1 we obtained some new results based on Eq. (3) [8–11]. Thus, for an inhomogeneous medium with spherical inclusions of regular structure for $\langle \varepsilon \rangle$ we obtained the following equation:

$$\langle \varepsilon \rangle = \varepsilon_1 \frac{\varepsilon_2 + 2\varepsilon_1 + f_2 (\varepsilon_2 - \varepsilon_1)}{\varepsilon_2 + 2\varepsilon_1 - 2f_2 (\varepsilon_2 - \varepsilon_1)} . \tag{5}$$

In the interval of change of the volume concentration of inclusions $0 \le f_2 \le \pi/6$ it agrees well with experiment [9]; however, for $\pi/6 \le f_2 \le 1$ it gives an incorrect result. This is due to the fact that formula (5) was derived with the intensity of the electric field inside the averaged medium calculated approximately. In the present work we are able to exclude this approximation.

According to [6], in determining $\langle \varepsilon \rangle$ of an inhomogeneous material from (4), the following equation is used for the averaged medium:

$$\vec{D}_{\text{m}} = \langle \vec{D} \rangle = \varepsilon_0 \langle \varepsilon \rangle \langle \vec{E} \rangle = \varepsilon_0 \vec{E}_{\text{m}} + \vec{P}_{\text{m}} . \tag{6}$$

If we use the definition of averaging (3) not only for the vectors $\vec{D}$ and $\vec{E}$ but also for the polarization vector, then for the mean polarization of an inhomogeneous material we obtain

$$\langle \vec{P}_{\text{in}} \rangle = \frac{1}{V_0} \int_{V_0} \vec{P} dV = \frac{1}{V_0} \sum_{k=1}^{n_0} \int_{V_{2k}} \vec{P}_{2k} \, dV . \tag{7}$$

We note that $\vec{P}_{2k} = 0$ outside V_{2k} while within it

$$\vec{D}_{2k} = \varepsilon_0 \, \varepsilon_2 \vec{E}_{2k} = \varepsilon_0 \vec{E}_{2k} + \vec{P}_{2k} . \tag{8}$$

A homogeneous averaged medium defined by Eq. (6) and an actual inhomogeneous medium with the mean polarization (7) are equivalent from the physical point of view [6], which, in particular, yields that

$\vec{P}_m$ of the effective medium and $\langle \vec{P}_{in} \rangle$ (7) of the inhomogeneous medium are equal. With Eq. (8) taken into account, the equality takes the form

$$(\varepsilon_m - 1)\, \varepsilon_0 \vec{E}_m = (\varepsilon_2 - 1)\, \varepsilon_0 \frac{1}{V_0} \sum_{k=1}^{n_0} \int_{V_{2k}} \vec{E}_{2k}\, dV .$$

This definition for $\langle \varepsilon \rangle = \varepsilon_m$ for constant parameters of the components of an inhomogeneous medium differs significantly from Maxwell's definition [3], and with approximate calculation of $\vec{E}_m$ it allowed one to derive formula (5).

The equivalence of the above-indicated media also implies the equality of the polarization energies of the averaged $\langle W_m \rangle = W_m$ and inhomogeneous $\langle W_{in} \rangle = W_{in}$ materials:

$$W_m = \frac{1}{2V_0} \int_{V_0} (\vec{E}_m \vec{D}_m - \vec{E}_0 \vec{D}_0)\, dV = \frac{1}{2V_0} \int_{V_0} (\vec{E}_{in} \vec{D}_{in} - \vec{E}_0 \vec{D}_0)\, dV = W_{in} . \tag{9}$$

Relation (9) holds for any values of the parameters of the media. We express these polarization energies of the fields in terms of the parameters of the inhomogeneous material and the effective medium, where the dielectric permeability of an inclusion and, consequently, of the averaged medium is a variable quantity. For this, we consider a dispersion medium with dielectric permeability $\varepsilon_1 = 1$, where an electric field of intensity $\vec{E}_0$ is established. In this medium, we introduce into a limited region of space, all at one time and not in separate portions, as was done by Bruggeman, foreign solid dielectric particles of volume V_{2k} with the dielectric permeability ε. With an infinitely small change in ε by $\delta\varepsilon$ the polarization energy will undergo an infinitely small change [7]

$$\delta W = -\frac{1}{2} \int \vec{E}_0\, \delta \vec{P} dV , \tag{10}$$

where the infinitely small change in the polarization vector is equal to

$$\delta \vec{P} = \delta\varepsilon \vec{E} = \delta\varepsilon\, (\vec{E}_0 + \delta \vec{E}) . \tag{11}$$

In (10), the polarization vector $\vec{P}$ outside the volumes of the dielectric particles is equal to zero. Expressing $\vec{E}_0 = \vec{E} - \delta \vec{E}$ from Eq. (10), with Eq. (11) taken into account, we obtain

$$\delta W = -\frac{1}{2} \int E^2\, \delta\varepsilon\, dV . \tag{12}$$

In the derivation of formula (12) the term $\vec{E} \delta\varepsilon \delta \vec{E}$ in relation to $E^2 \delta\varepsilon$ is assumed to be equal to zero as an infinitely small quantity of second order. Integrating (12) within the limits from $\varepsilon = \varepsilon_1 = 1$ to ε_2, for the polarization energy n_0 of the inclusions inserted into the region of averaging V_0 we have

$$W_{in} = -\frac{1}{2} \sum_{k=1}^{n_0} \int_{V_{2k}} \int_{\varepsilon_1}^{\varepsilon_2} E_{2k}^2\, d\varepsilon dV , \tag{13}$$

where $\vec{E}_{2k} = \vec{E}_{2k}(\varepsilon)$ is the intensity inside the k-th inclusion, which depends on the dielectric permeability ε. We note that in expression (13) the volume integral is extended only inside the inclusions. And for the polarization energy of the averaged medium, whose dielectric permeability is a variable quantity and varies from $\varepsilon = \varepsilon_1 = 1$ to $\varepsilon = \varepsilon_m$, similarly to (13) we obtain

$$W_m = -\frac{1}{2} \int_{V_0} \int_{\varepsilon_1}^{\varepsilon_m} E_m^2 \, d\varepsilon \, dV , \tag{14}$$

where $\vec{E}_m = \vec{E}_m(\varepsilon)$ is the intensity inside the averaged medium, which depends on the dielectric permeability ε.

Equating (13) and (14), we finally obtain the relation

$$-\frac{1}{2} \int_{V_0} \int_{\varepsilon_1}^{\varepsilon_m} \vec{E}_m^2 \, d\varepsilon \, dV = -\frac{1}{2} \sum_{k=1}^{n_0} \int_{V_{2k}} \int_{\varepsilon_1}^{\varepsilon_2} \vec{E}_{2k}^2 \, d\varepsilon \, dV , \tag{15}$$

which is equivalent to (9).

Analysis of expression (15) shows that it satisfies all the requirements on the theory of investigation of inhomogeneous materials. First, according to the theory of averaging, this relation is derived without any approximations or assumptions. It results from the equivalence of the model suggested according to (6) and the real inhomogeneous medium. It will be shown below that this very general integral relation makes it possible to obtain, in the limiting cases of the concentrations of fillings, physically correct results that agree well with the experimental data of the available investigations.

We also note that the equality for polarization energy (15) is applicable for inclusions of any geometric shape, since relation (10) is independent of the shape of the dielectric body. Moreover, although it is assumed in deriving (15) that the number of inclusions n_0 inside the averaging region V_0 is bounded, it can be rather large. Therefore, formula (15) is also applicable for concentrated inhomogeneous materials. In particular cases Eq. (15) yields presently known basic equations for ε_m.

Thus, to calculate the mean parameter ε_m of an inhomogeneous material, it is necessary to calculate the intensities inside an inclusion $\vec{E}_{2k}(\varepsilon)$ and the averaged homogeneous medium $\vec{E}_m(\varepsilon)$ of the model.

Below, as an example, we consider basic matrix two-component inhomogeneous materials in which the relative dielectric permeability of the inclusions is a variable quantity and can vary in the range $1 \le \varepsilon \le \infty$. The external field with the intensity $\vec{E}_0$ is assumed to be electrostatic and homogeneous.

For an inhomogeneous material of regular structure with ellipsoidal inclusions of low concentration identically oriented along the x axis and also for concentrated mixtures of periodic structure in the presence of the above-indicated external field directed along the x axis, foreign particles are polarized identically and uniformly. Then, relation (15) takes the form

$$\int_{\varepsilon_1}^{\varepsilon_m} \vec{E}_{mx}^2 \, d\varepsilon = f_2 \int_{\varepsilon_1}^{\varepsilon_2} \vec{E}_{2kx}^2 \, d\varepsilon . \tag{16}$$

We note that for the inhomogeneous materials considered above, the parameters that characterize the field and the medium are independent of the ordinal number k of the inclusions; therefore it will be omitted in subsequent formulas.

First, we consider the case $f_2 << 1$ where it is possible to ignore the interaction between foreign inclusions. The foreign particles here are polarized under the influence of the external field. Then, according to the superposition method, for $\vec{E}_{2x}$ we have

$$\vec{E}_{2x} = \vec{E}_{0x} + \vec{E}_{\text{pol}x} = \vec{E}_{0x} - \frac{N_x \vec{P}_{2x}}{\varepsilon_0}. \tag{17}$$

Substituting $\vec{P}_{2x}$ from (8) into (17), for $\vec{E}_{2x}$ at $\varepsilon_2 = \varepsilon$ we obtain

$$\vec{E}_{2x}(\varepsilon) = \frac{\vec{E}_{0x}}{1 + (\varepsilon - 1) N_x}, \tag{18}$$

where ε varies from $\varepsilon = 1$ to $\varepsilon = \varepsilon_2$.

Similarly, for $\vec{E}_{\text{m}x}$ we write

$$\vec{E}_{\text{m}x}(\varepsilon) = \frac{\vec{E}_{0x}}{1 + (\varepsilon - 1) N_x}, \tag{19}$$

where ε of the homogeneous medium varies from 1 to $\varepsilon = \varepsilon_\text{m}$.

Substituting (18) and (19) into (16) and integrating, after simple algebra we obtain a formula for the mean dielectric permeability of an inhomogeneous medium with ellipsoidal inclusions:

$$\varepsilon_\text{m} = \frac{1 + (\varepsilon_2 - 1)\,[f_2 + (1 - f_2) N_x]}{1 + (\varepsilon_2 - 1)(1 - f_2) N_x}. \tag{20}$$

We note that for $N_x = 1/3$ (spherical inclusions) formula (20) is converted into Maxwell–Lorenz's formula [1–4] and for $N_x = 1/2$ (cylindrical inclusions) into Rayleigh's equation [2]. The first approximation of (20) is Fricke's formula [2]. Indeed, expanding (20) about $f_2 = 0$ in a series in powers of f_2 and restricting ourselves to the term linear in f_2, from (20) we obtain Fricke's formula:

$$\varepsilon_x = 1 + \frac{f_2(\varepsilon_2 - 1)}{(\varepsilon_2 + 2) N_x}, \tag{21}$$

which for $N_x = 1/3$ is converted into Landau–Lifshits's formula [7].

We now consider the investigated medium for high concentrations, when the interaction between inclusions is to be taken into account. In such cases these particles (equivalent dipoles) are polarized (see (3)) under the effect of the field acting on a particle; it is called the Lorenz field. The averaged intensity of the Lorenz field along the x axis, with Eq. (7) taken into account, is

$$\langle \vec{E}_{\text{a}x} \rangle = \frac{1}{V_0} \int\limits_{V_0} \left(\vec{E}_{0x} + \frac{N_x \vec{P}}{\varepsilon_0} \right) dV = \vec{E}_{0x} + \frac{f_2 \vec{P}_{2x} N_x}{\varepsilon_0}. \tag{22}$$

Then, replacing $\vec{E}_{0x}$ in Eq. (17) by the Lorenz field (22), for $\vec{E}_{2x}$ we obtain

$$\vec{E}_{2x} = \frac{\vec{E}_{0x}}{1 + (\varepsilon - 1)(1 - f_2) N_x}. \tag{23}$$

Now, we calculate the intensity of the field $\vec{E}_{\text{m}x}$ inside the averaged medium defined by Eq. (6). The averaged value of the field along the x axis acting on a dipole is now equal to

$$\langle \vec{E}_{\text{a}x} \rangle = \langle \vec{E}_{0x} \rangle + \langle \vec{P}_{\text{m}x} \rangle \frac{N_x}{\varepsilon_0}. \tag{24}$$

From the condition of the equivalence of the averaged medium to the actual inhomogeneous material we obtain the equality of the polarization vectors defined by (6) and (7). From this equality for an inhomogeneous material of periodic structure we have

$$\vec{P}_{\mathrm{m}x} = \frac{1}{V_0} \sum_{k=1}^{n_0} \int_{V_{2k}} \vec{P}_{2x}\, dV = f_2 \vec{P}_{2x} \,. \tag{25}$$

Then relation (24), with (25) taken into account, will take the form

$$\langle \vec{E}_{\mathrm{a}x} \rangle = \vec{E}_{0x} + \frac{N_x f_2^2\, \vec{P}_{2x}}{\varepsilon_0} = \vec{E}_{0x} + \frac{N_x f_2 \vec{P}_{\mathrm{m}x}}{\varepsilon_0} \,. \tag{26}$$

Determining $\vec{P}_{\mathrm{m}x}$ from Eq. (6) and representing the intensity of the field inside the effective medium as a superposition of the Lorenz field (26) and the field of polarizations

$$\vec{E}_{\mathrm{m}x} = \langle \vec{E}_{\mathrm{a}x} \rangle - \frac{N_x \vec{P}_{\mathrm{m}x}}{\varepsilon_0} \,, \tag{27}$$

for $\vec{E}_{\mathrm{m}x}$ we have

$$\vec{E}_{\mathrm{m}x} = \frac{\vec{E}_{0x}}{1 + (\varepsilon - 1)\,(1 - f_2)\, N_x} \,, \tag{28}$$

where ε of the homogeneous medium varies within $1 \le \varepsilon \le \varepsilon_{\mathrm{m}}$.

Substituting (23) and (28) into formula (16), for $\varepsilon_{\mathrm{m}x} = \varepsilon_{\mathrm{m}}$ with $\varepsilon_1 \ne 1$ we now obtain

$$\langle \varepsilon_x \rangle = \varepsilon_{\mathrm{m}} = \varepsilon_1 \frac{\varepsilon_1 + (\varepsilon_2 - \varepsilon_1)\,[f_2 + (1 - f_2)^2 N_x]}{\varepsilon_1 + (\varepsilon_2 - \varepsilon_1)\,(1 - f_2)^2 N_x} \,. \tag{29}$$

When $N_x = 1/3$ Eq. (29) yields a formula for an inhomogeneous material with spherical inclusions:

$$\varepsilon_{\mathrm{m}} = \varepsilon_1 \frac{\varepsilon_2 + 2\varepsilon_1 + f_2\,(\varepsilon_2 - \varepsilon_1)\,(1 + f_2)}{\varepsilon_2 + 2\varepsilon_1 - f_2\,(\varepsilon_2 - \varepsilon_1)\,(2 - f_2)} \,, \tag{30}$$

and for a mixture with cylindrical inclusions:

$$\varepsilon_{\mathrm{m}} = \varepsilon_1 \frac{\varepsilon_2 + \varepsilon_1 + (\varepsilon_2 - \varepsilon_1)\, f_2^2}{\varepsilon_2 + \varepsilon_1 - f_2\,(\varepsilon_2 - \varepsilon_1)\,(2 - f_2)} \,. \tag{31}$$

Analysis of formulas (29)–(31) shows that they satisfy the above general requirements on the results of investigations carried out in the theory of inhomogeneous materials. The first factor is that the indicated equations were derived from the condition of equivalence between the suggested model and the actual inhomogeneous medium (15). Another important factor is that the intensities $\vec{E}_2$ and $\vec{E}_{\mathrm{m}}$ were calculated according to the principal definition of averaging of an inhomogeneous material. Finally, formulas (29)–(31) give correct results for the limiting concentrations of the inclusions. Thus, the indicated formulas yield $\varepsilon_{\mathrm{m}} = \varepsilon_1$ for $f_2 = 0$ and $\varepsilon_{\mathrm{m}} = \varepsilon_2$ for $f_2 = 1$.

TABLE 1. Relative Dielectric Permeability Calculated from Formula (30) and Comparison with Existing Formulas

f_2	⟨ε⟩, experiment	Formulas					
		1	2	3	4	5	6
0.05	2.3170	2.3165	2.3168	2.3149	2.3154	2.3177	2.3170
		–5	–2	–21	–16	+7	0
0.10	2.4110	2.4074	2.4085	2.4116	2.4026	2.4124	2.4120
		–36	–25	–6	–84	+14	+10
0.15	2.5110	2.5006	2.5032	2.5077	2.4901	2.5124	2.5106
		–92	–78	–33	–209	+14	+6
0.20	2.6110	2.5968	2.6009	2.6072	2.5750	2.6183	2.6138
		–142	–101	–38	–360	+73	+28
0.25	2.7140	2.6954	2.7017	2.7099	2.6618	2.7306	2.7213
		–186	–123	–41	–762	+166	+73
0.30	2.8240	2.7968	2.8055	2.8181	2.7522	2.8496	2.8330
		–272	–185	–59	–718	+258	+90
0.35	2.9490	2.9012	2.9124	2.9255	2.8396	2.9766	2.9480
		–478	–366	–235	–1094	+276	–10

Note. Comparison of formulas: 1) Maxwell–Lorenz, 2) Bruggeman–Hanai, 3) Odelevskii–Kondorskii, 4) Landau–Lifshits, 5) formula (5), 6) formula (30) with experimental data obtained by Reynolds.

Now, we will show that these formulas agree better with experiment than those from existing works. For this, we use experimental data available in the literature for an inhomogeneous material with spherical and cylindrical inclusions.

Table 1 presents results of calculations by formula (30) for ε_1 = 2.228 (carbon tetrachloride) and ε_2 = 4.594 (spherical glass particles) within the limits of volume concentration $0 \le f_2 \le 0.35$. Also presented there are Reynolds experimental data for an inhomogeneous material with the above-indicated values of ε_1 and ε_2. Deviations of the theoretical data (multiplied by 10^4) are indicated, where the plus sign denotes values that exceed the experimental ones, and the minus sign ones that are lower.

The data of Table 1 show that formula (30) is in better agreement with experiment than the Maxwell–Lorenz, Odelevskii–Kondorskii [2], Bruggeman–Hanai, Reynolds–Hugh, and Landau–Lifshits formulas [7].

Close agreement of Eq. (30) with experiment is also the case where the inclusions are conducting spherical particles. This can be seen from the data of Table 2, which were obtained within $0 \le f_2 \le 0.25$ for an inhomogeneous material whose matrix medium is viscous oil (ε_1 = 2.10) and whose spherical inclusions are made of iron ($\varepsilon_2 = \infty$) [1]. It is evident from these data that formula (30) agrees more accurately with experiment than the formulas derived by Hering, Fokin and Shermergor, Maxwell and Lorenz, and Landau and Lifshits [1–7]. We note that, whereas the minimum and maximum deviations from experimental data for these formulas are 11.9 and 42%, 11.9 and 32.8%, 6.5 and 32.8%, 7 and 41%, respectively, for formula (30) they are 5 and 21.5%.

Table 3 contains experimental data for an inhomogeneous material with cylindrical inclusions where the external homogeneous field is directed normal to the axes of identically oriented inclusions. The length of these particles with a dielectric permeability ε_2 = 7.034 is equal to $8 \cdot 10^{-5}$ m, while the diameter is equal to 10^{-5} m. The matrix medium is a medium with a dielectric permeability ε_1 = 2.228 [1]. The table also contains results for ε_m calculated from Rayleigh's formula [1, 2] and formula (31). The data of the table indicate that the accuracy of formula (31) is higher than that of Rayleigh's formula. As the volume concentration increases, the discrepancy between the Rayleigh formula and experiment increases, and within the limits $0 \le f_2 \le 0.4$ it becomes very appreciable. At the same time, the results of calculations by formula (31) agree rather well with experiment.

TABLE 2. Relative Dielectric Permeability Calculated from Formula (30) and Comparison with Existing Formulas

f_2	$\langle\varepsilon\rangle$, experiment	Formulas					
		1	2	3	4	5	6
0.05	2.60	2.29	2.29	2.43	2.41	2.45	2.47
0.10	3.30	2.52	2.66	2.80	2.73	2.88	2.88
0.15	4.00	2.81	3.05	3.21	3.09	3.45	3.84
0.20	4.90	3.17	3.55	3.67	3.36	4.20	4.07
0.25	6.25	3.61	4.20	4.20	3.67	5.25	4.90

Note. Comparison of formulas: 1) Hering, 2) Fokin–Shermergor, 3) Maxwell–Lorenz, 4) Landau–Lifshits, 5) formula (5), 6) formula (30) with experimental data obtained by Reynolds.

TABLE 3. Relative Dielectric Permeability Calculated from Formula (31) and Comparison with Rayleigh's Formula

f_2	$\langle\varepsilon\rangle$, experiment	Formulas	
		1	2
0.10	2.59	2.61	2.48
0.20	2.92	2.74	2.80
0.30	3.34	3.06	3.17
0.40	3.82	3.39	3.61

Note: Comparison of: 1) Rayleigh's formula and 2) formula (31) with experimental data obtained by Reynolds.

Unfortunately, the authors are unaware of experimental data for an inhomogeneous material with ellipsoidal inclusions, and for this reason formula (29) was not fitted to experimental data and was not compared with other existing formulas. However, it was clearly established that the accuracy of (29) is higher than that of Fricke's formula because the latter is a first approximation of (29).

In conclusion we note that in particular cases, based on the intensities inside inclusions and the effective medium known from the existing works, from relation (15) we obtain formulas derived by Maxwell, Lorenz, Rayleigh, Maxwell (for a two-layer dielectric with series-connected layers), Wiener (for a lamellar structure with layers connected in parallel), Wagner, Landau and Lifshits, Kondorskii and Odelevskii, etc.

Thus, in the present work a method is suggested for calculating the averaged parameters of inhomogeneous materials with variable parameters of the inclusions. Since this method, which is based on the theory of averaging, satisfies the general requirements on the existing investigations, it allows one to obtain new formulas to calculate the averaged parameters of inhomogeneous media.

NOTATION

$\vec{E}_{\rm m}$ and $\vec{D}_{\rm m}$, vectors of the electric field inside of the averaged medium; $\vec{P}_{\rm m}$, polarization vector of the averaged medium; ε_0, electric constant; V_0, physically infinitely volume of averaging; n_0, number of inclusions in the volume of averaging; V_{2k}, volume of the k-th inclusion; $\vec{E}_{2k}$, intensity inside the k-th particle; $\vec{E}_{\rm in}$ and $\vec{D}_{\rm in}$, vectors of the electric field inside the inhomogeneous medium; $\vec{E}_0$ and $\vec{D}_0$, vectors of the electric field in vacuum; f_2, filling concentration of inclusions; $\vec{E}_{{\rm p}x}$, intensity of the polarization field along the x axis; N_x, coefficient of depolarization of ellipsoidal particles along the x axis. Subscripts: m, medium; in, inhomogeneous; a, acting; p, polarization.

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